Transition from intermittency to periodicity in lag synchronization in coupled Rössler oscillators

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The dynamical and statistical behavior of lag synchronization in two coupled self-sustained chaotic Rössler oscillators is reexamined. The lack of uniqueness in the conventional characterization of lag synchronization based on the similarity function has caused much skepticism about the existence of lag synchronization. We provide an evidence that the emergence of lag synchronization is associated with the transition from on-off intermittency to a periodic structure in the laminar phase distribution.

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Recently, chaos synchronization in coupled systems has attracted great attention and has been extensively studied. One of the most important motivations is to understand the coherent dynamical behavior of the coupled systems. Several typical synchronizations have been identified as complete (or full) synchronization [1–4], generalized synchronization [5], phase synchronization [6], and lag synchronization [7,8] and they represent the difference in the degree of correlation between interacting systems. Among these synchronizations, complete synchronization is the strongest in the degree of correlation and describes the interaction of two identical systems, leading to their trajectories remaining identical in the course of temporal evolution, i.e., $\mathbf{x}_1(t)=\mathbf{x}_2(t)$. Generalized synchronization, as introduced for drive-response systems, is defined as the presence of a functional relationship between the states of the responser and driver, i.e., $\mathbf{x}_1(t)=f(\mathbf{x}_2(t))$. Phase synchronization describes the identity in the phase of nonidentical chaotic oscillators, whereas their amplitudes may remain chaotic and uncorrelated. Lag synchronization has been proposed as the coincidence of the states of two coupled systems in which one of the system is delayed by a finite time τ, i.e., $\mathbf{x}_1(t)=\mathbf{x}_2(t+\tau)$. More recently, a generalized time-lagged chaotic synchronization has been observed in two unidirectionally coupled Chua's circuits in the presence of large parameter mismatches [9]. Most recently, a discussion on the relationship and definition of synchronization has been reported [10].

There is a pressing need to further clarify the basic definition and fundamental concept of chaos synchronization [10,11] as the field is expanding rapidly and is driven by its success in a variety of science and engineering applications [12]. One typical problem concerns the realizability of the lag synchronization as originally defined by the vanishing of a similarity function [7]. Unlike the robust equality relation in a complete synchronization, which obviously is a solution of the coupled systems, the identity of two chaotic oscillators with a time lag seems to be incredible and uncertain. As a result, a similarity function has rarely been found to be identically zero and some researchers have proposed to use an approximate equality to express the time-lagged synchronous relation [7,10]. Obviously, a more precise characterization of lag synchronization is desired, as the concept has been of much interest to many investigators [8]. The objective of the present paper is to explore the dynamical and statistic properties of lag synchronization and to identify a transition from intermittency to periodicity.

We adapt the systems of two Rössler oscillators that are coupled with some parameter mismatch [7],

$$\begin{aligned}\dot{x}_{1,2}&=-\omega_{1,2}y_{1,2}-z_{1,2}+\varepsilon(x_{2,1}-x_{1,2}),\\ \dot{y}_{1,2}&=\omega_{1,2}x_{1,2}+ay_{1,2},\\ \dot{z}_{1,2}&=f+z_{1,2}(x_{1,2}-c).\end{aligned}\tag{1}$$

We choose the same set of parameters as used by the original authors of the first paper on lag synchronization [7], i.e., $a=0.165$, $f=0.2$, $c=10$, and $\omega_{1,2}=\omega_0\pm\theta$ ($\omega_0=0.97$, $\theta=0.02$, $\omega_1=0.99$, and $\omega_2=0.95$). Here, 2θ and ε determine the mismatch of the natural frequencies and coupling strength, respectively.

To characterize lag synchronization, Rosenblum *et al.* [7] introduced a similarity function $S(\tau)$ as a time averaged difference between the variables x_1 and x_2 (with mean values being subtracted) with a time delay τ,

$$S(\tau)=\sqrt{\frac{\langle[x_2(t+\tau)-x_1(t)]^2\rangle}{[\langle x_1^2(t)\rangle\langle x_2^2(t)\rangle]^{1/2}}}\geqslant 0.\tag{2}$$

The minimum of $S(\tau)$ is denoted as $\sigma=\min_\tau S(\tau)$ [we denote the corresponding lag time as ϕ, i.e., $\sigma=S(\phi)$]. Certainly, if the signals $x_1(t)$ and $x_2(t)$ are independent, the difference between them is of the same order as the signals themselves, i.e., $S(\tau)\approx\sqrt{2}$ for all τ. If $x_1(t)=x_2(t)$, as the case of complete synchronization, $S(\tau)$ reaches its minimum $\sigma=0$ for $\tau=0$. When $S(\tau)$ has a minimum value for nonzero time shift τ, indicating a time lag between the two processes, lag synchronization occurs. This condition has been extensively used to identify the transition to lag synchronization [7,8].

The minimum of the similarity function σ is depicted in Fig. 1, together with the corresponding lag time ϕ, versus the coupling ε. At first glance, it appears that the value of σ drops to zero with ϕ being a nonzero value at the critical coupling strength $\varepsilon_c=0.15$. In fact, Rosenblum and coworkers have described the following scenario to synchronization for this kind of coupled systems: with the increase in

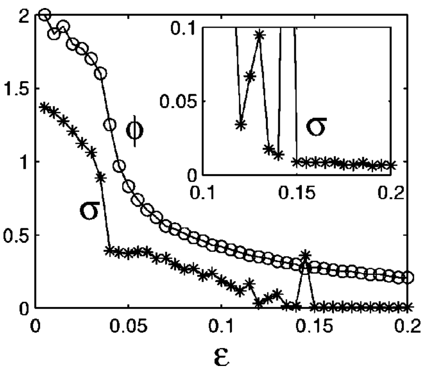

FIG. 1. The minimum of the similarity function σ and the corresponding lag time ϕ vs the coupling strength ε labeled by star and circle points, respectively. The zoomed-up part of σ vs ε is displayed in the enlargement.

the coupling strength, phase synchronization occurs before lag synchronization, and finally, complete synchronization takes place. However, a careful examination as shown in the enlargement of Fig. 1 indicates that the minimum of the similarity function σ does not really vanish even at a very large coupling strength ε ($\sigma \approx 6\times 10^{-3}$ when $\varepsilon=0.2$). It is seen that only an approximate relation $x_1(t)\approx x_2(t+\phi)$ exists. In fact, for any two chaotic time series, one can always utilize the similarity function defined above to choose an optimum lag time corresponding to the minimum of the similarity function, and obtain an approximate equality between $x_1(t)$ and $x_2(t+\phi)$. So it seems that the concept of lag synchronization given by the relation $x_1(t)=x_2(t+\phi)$ is very sound. However, one may wonder whether lag synchronization defined by the minimum of the similarity function (zero) can really be realized. If this is the case, is there any other intrinsic feature associated with this chaos synchronization? This problem is addressed in the rest of this paper.

First, we choose the coupling strength $\varepsilon=0.1$ below the critical value $\varepsilon_c=0.15$, and plot $x_1(t)$ vs $x_2(t)$ and $x_2(t+\phi)$ in Figs. 2(a) and 2(b), respectively. The optimized lag time ϕ is 0.42. It is clear that $x_2(t+\phi)$ is not equal to $x_1(t)$ even at the optimized lag time ϕ. However, the time-shifted plot [Fig. 2(b)] does show a more concentrated distribution around the diagonal. In Fig. 2(c), the difference of $x_1(t)-x_2(t+\phi)$ is plotted against time. Obviously, the difference exhibits typical feature of on-off intermittency [13], which usually appears at the chaos desynchronization of the coupled identical systems with the "off" state near the laminar phase state and the "on" state showing random bursts. It is interesting to note that except for the irregularity in time, the amplitude of bursts is very large and is of the order of system states. To analyze the statistical feature of this irregular motion, we use the technique widely used in the statistical analysis of intermittency [13] and compute numerically the distribution of laminar phases denoted by the amplitude less than a threshold value $\Delta=2.0$. A universal asymptotic $-3/2$ power-law distribution is observed in Fig. 2(d) for these coupled nonidentical oscillator systems. This is quite typical for on-off intermittency. A similar result is obtained for a smaller threshold value $\Delta=1.0$.

Next we choose the coupling strength above the critical value. For such a coupling strength, some change in coupled states is expected due to lag synchronization. As shown in Fig. 3, we set $\varepsilon=0.2$ and carry out our computation in the same manner as in Fig. 2. Indeed, the states of two systems show a good approximate relation, $x_1(t)\approx x_2(t+\phi)$ with ϕ

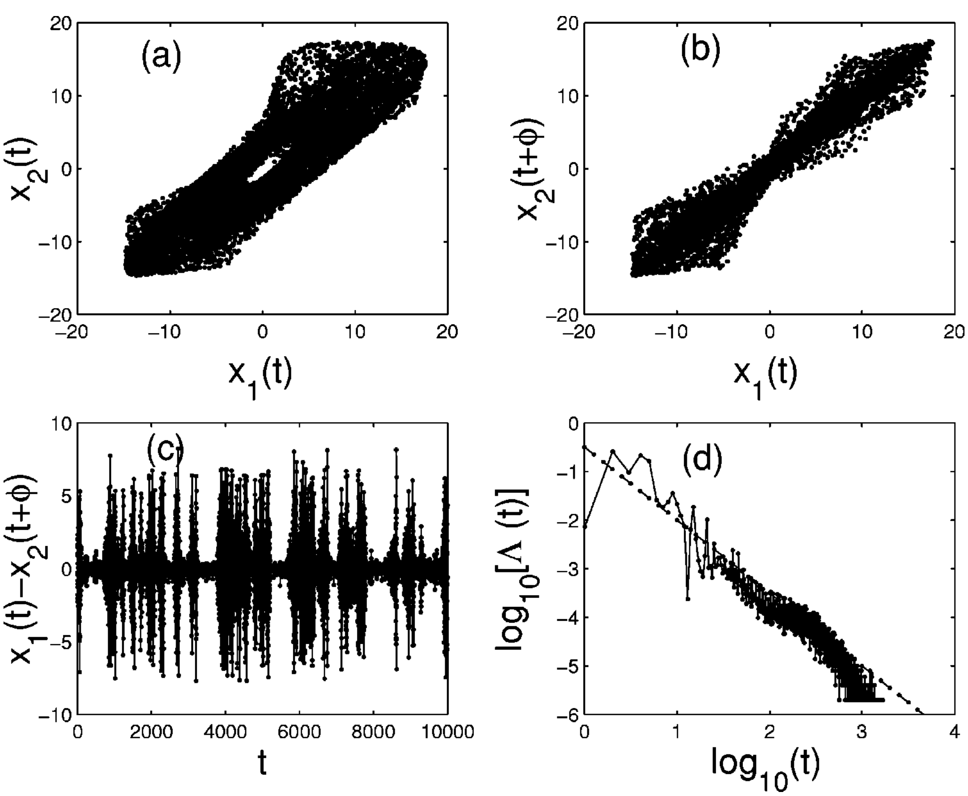

FIG. 2. A test case at $\varepsilon=0.1$, below the critical coupling strength $\varepsilon_c=0.15$. (a), (b) $x_2(t)$ and $x_2(t+\phi)$ vs $x_1(t)$ are plotted, respectively ($\phi=0.42$). (c) The on-off intermittency of $x_1(t)-x_2(t+\phi)$ vs t. (d) The statistical distribution of laminar phases that shows the $-3/2$ power-law scaling. The threshold value used to compute the laminar phases is $\Delta=2.0$.

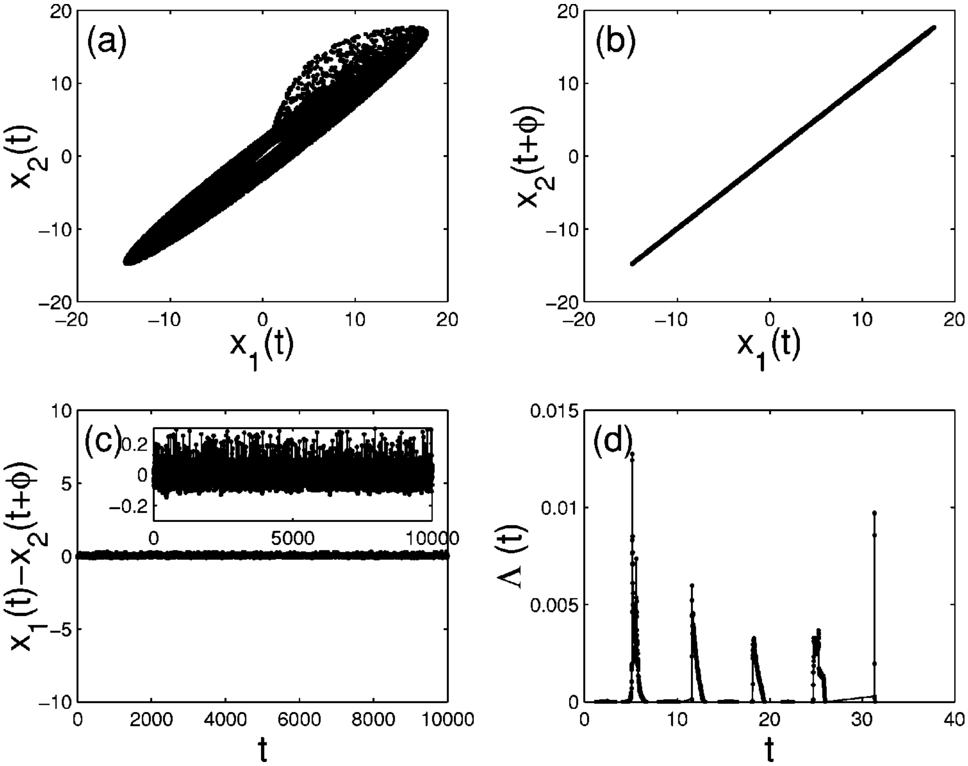

FIG. 3. (a)–(d) The same as Fig. 2, with $\varepsilon=0.2$, above $\varepsilon_c=0.15$ ($\phi=0.21$, $\Delta=0.1$). The amplitude of $x_1(t)-x_2(t+\phi)$ is squeezed to the regime of $(-0.1,0.3)$ showing $x_1(t)\approx x_2(t+\phi)$. The periodic distribution shows the rotational period of a single Rössler oscillator.

$=0.21$. The distribution is localized entirely at the diagonal [Fig. 3(b)] and the difference of states is much squeezed to the line of zero [Fig. 3(c)]. However, the difference does not vanish precisely [see the enlarged plot in Fig. 3(c)] and it fluctuates in the neighborhood of $(-0.1,0.3)$, which further illustrates the unsuitability of using the similarity function for the characterization of lag synchronization.

In contrast to the appearance of Fig. 2(c), the distribution in Fig. 3(c) becomes more regular and is much smaller in amplitude. If the lag-synchronized state is really a unique physical state, it should exhibit a *unique feature* in the distribution of the difference of states below and above the lag synchronization. In Fig. 3(d), we plot the distribution of the "laminar phase" against the survival time in the same manner as that in Fig. 2(d). Instead of the chaotic distribution, we observe a distinct periodic structure in the distribution of the "laminar phase" by means of a small threshold value $\Delta=0.1$. The periodic distribution is characterized by $t=nT, n=1,2,\ldots,$ where T is about the rotational period of a single Rössler oscillator. It should be pointed out that the periodicity is associated with statistic distribution, while the motions of the coupled oscillators are still chaotic.

This regularity under the seemingly irregular minor mismatch of two oscillators is dramatically different from the turbulent behavior of on-off intermittency observed when the coupling strength is far below the critical value. It is found that lag synchronization leads to the transition from on-off intermittency to periodic bursts, and such a transition gives rise to the sharp reduction in the similarity function. This transition thus uniquely signals the emergence of lag synchronization.

In an attempt to elucidate lag synchronization further, we choose the coupling strength $\varepsilon=0.14$, which is slightly smaller than the critic point $\varepsilon=0.15$. Results are shown in Fig. 4. The trajectory is distributed essentially around the diagonal with some nonideal dots as depicted in Fig. 4(a). The difference of states as shown in Fig. 4(b) indicates the co-existence of some large bursts similar to that in Fig. 2(c) and some small ones similar to that in Fig. 3(c). Figure 4(c) shows the log-log plot of the distribution of the "laminar phase" with a threshold value of $\Delta=0.5$ to catch the statistical behavior of large bursts. In the distribution the universal asymptotic $-3/2$ power law for on-off intermittency in the small time part and the large deviation in its shoulder at large times are both observed. In contrast, if a small amplitude threshold value $\Delta=0.2$ is used in Fig. 4(d) to compute the distribution of "laminar phase," a periodic structure is observed. Figures 4(c) and 4(d) clearly display the precursor of emergence of lag synchronization: a transition from power law to a regular periodic distribution.

We have thus confirmed the following scenario for the onset of lag synchronization of chaotic oscillators. When the coupling strength is much less than the critical point, the "laminar phase" distribution of the difference of states gives the signature of on-off intermittency, the power-law scaling. As the coupling strength is increased up to the critical point, the "laminar phase" distribution does not satisfy the power-law scaling and the on-off intermittency disappears. Instead, the periodic structure dominates the "laminar phase" distribution [Fig. 3(d)]. Clearly, such change shows the emergence of lag synchronization.

It remains to verify that the critic coupling strength $\varepsilon_c=0.15$ is a transition point in the sense of on-off intermittency. In statistical analysis, the critical exponent of -1 in the power-law scaling of the mean laminar phase as a function of deviation from critical onset parameter is one of im-

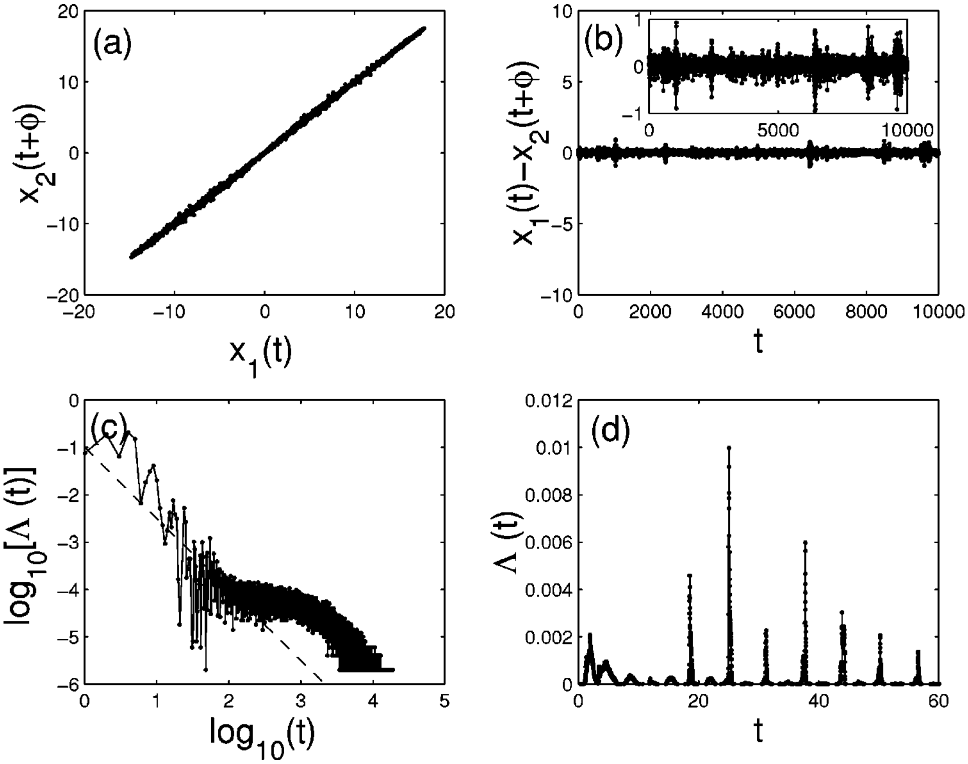

FIG. 4. (a)–(d) The same as Fig. 2, with $\varepsilon=0.14$, below and very close the critical coupling ($\phi=0.30$). The large fluctuation of on-off intermittency is much squeezed. Threshold values for laminar phases are $\Delta=0.5$ for (c) and $\Delta=0.2$ for (d), respectively.

portant features [13] for the onset of on-off intermittency. Figure 5 shows the numerically determined mean laminar phase as the function of $\delta=\varepsilon_c-\varepsilon$ for the on-off intermittency behavior before lag synchronization. We use the technique as in Ref. [13]. The quantity plotted is $L(t)-c_0$, where $L(t)$ is mean laminar phase and c_0 is determined from a least-squares fit of the theoretical model $L(t)=c_0+c_1/\delta$. The fit gives parameters $c_0=-223.3$ and $c_1=9.2$. This case clearly shows a power-law scaling of the mean laminar phase for ε near ε_c, with a critical exponent of -1 indicating the transition. Meanwhile, the critical value of the coupling strength $\varepsilon_c=0.15$ is also confirmed.

A discussion of the common feature between lag synchronization and complete synchronization is in order. It is well known that the complete synchronization happens under one of the following two conditions. The first is the case where there is an invariant manifold of chaos synchronous states for identical systems without parameter mismatch with a finite coupling strength at a finite time. Second, complete synchronization can also occur for nonidentical systems with infinite coupling strength. It is believed that complete synchronization is typically characterized by the vanishing of on-off intermittency of the laminar phase distribution [3,4]. However, the present paper shows that there is an intermediate transition in the laminar phase distribution, i.e., the transition from on-off intermittency to periodic bursts. We propose to use this phenomenon as one of key features for an identification of the onset of lag synchronization. Obviously these periodic bursts will completely disappear as the coupling strength increases to the infinity.

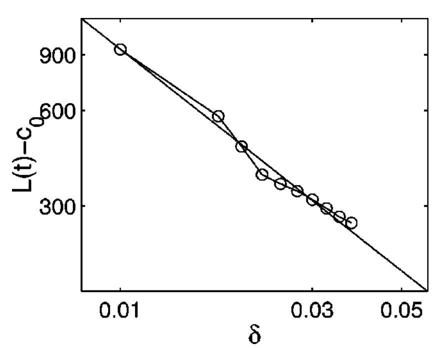

FIG. 5. A log-log plot of $L(t)-c_0$ vs δ ($\delta=\varepsilon_c-\varepsilon$, $\varepsilon_c=0.15$). The critical exponent of -1 in the power-law scaling confirms the on-off intermittent behavior before the onset of lag synchronization and the existence of the critical coupling strength $\varepsilon_c=0.15$.

In conclusion, we have studied the statistical properties of lag synchronization in two coupled chaotic Rössler oscillators with small rotational parameter mismatch. The conventional description based on the similarity function is neither rigorous nor unique, and leads to some confusion in the characterization of lag synchronization. The present investigation reveals that the transition from on-off intermittency to a periodic structure in the laminar phase distribution is one of the main features of this chaos synchronization. To our knowledge, this is the first observation of such a periodic distribution.

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